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FREE-SURFACE PROFILE OF A JET FLOW IN U-SHAPED CHANNEL WITHOUT GRAVITY EFFECTS

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Abstract. This paper concern with the two dimensional, steady jet flow problem of inviscid and incompressible fluid in semi infinite channel of the form U. This problem is solved by applying two methods. One is explicitly and analytically utilize the hodograph-transformation. The other is numerically using the series truncation method. The obtained results showed a good agreement between them and the comparison of these surface chapes is illustrated.

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1. Introduction

A jet flow can be seen in many engineering problems and flow in semi infinite tube is one example of these problems. This kind of problems is difficult to solve resolve because of the shape of the free surface which is unknown and the characterization of the jet by a nonlinear condition in the free boundary.

In this work we present the solution of flow in semi infinite channel by assuming that the influence of gravity relative to inertia is negligible. The fluid is assumed to be inviscid, incompressible and the flow is irrotationnal.

The mathematical solution can be obtained exactly via the free stream line theory due to Birkhoff [6] based on the hodograph transformation and numerically by using the series truncation method by many authors. Vanden-Broeck Jean-Marc [8], A. Gasmi and H. Mekias [1] and A. Gasmi [2] and [3], etc....

For each value of the width of the tube, we found that there is a solution of this problem using both methods considered. These results showed a good agreement between them and the representation of the surface profile is illustrated.

The problem is formulated in Section 2 and the exact solution is presented in section 3. The the numerical scheme, the results and the form of the free surface are done in section 4. Finally, we give a conclusion in Section 5.

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2. Mathematical formulation

The irrotationnal flow in semi-infinite rectangular channel is considered. the fluid is assumed to be inviscid and incompressible. The effects of gravity is neglected. Far upstream the flow is uniform with a velocity U and amplitude H. The flow is limited by the walls of the channel $A'B'$, $B'B$ BA and the free surface. One chooses as coordinates the wall $A'B'$ on x-axis and the y-axis is perpendicular and passing by B' (see Figure 1.). We introduce the

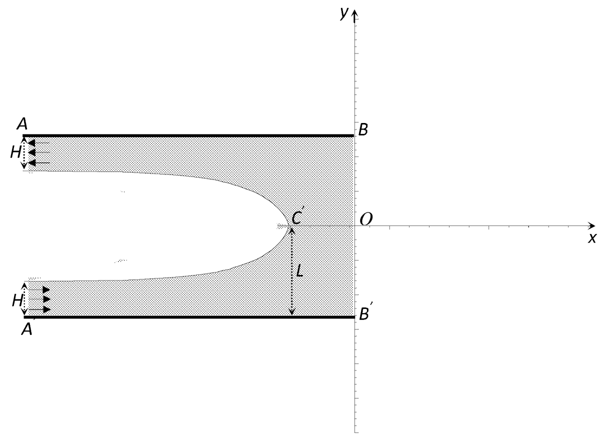

Figure 1. The x-plane

complex velocity potential $f = \varphi + i\psi$. By this the domain occupied by the fluid in the x-plane can be transformed to an infinite band (see Figure 2).

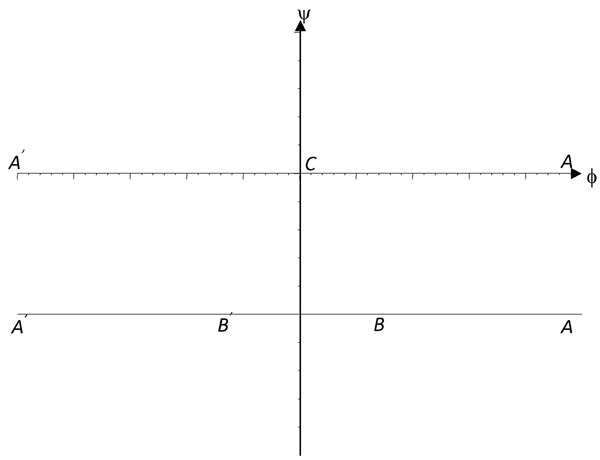

Figure 2. The f-plane

The mathematical problem is to determine the variable φ who verifies the following conditions:

(1) $$\Delta\varphi = 0,$$

in the interior of the flow filed.

(2) $$(\frac{\partial \varphi}{\partial x})^2 + (\frac{\partial \varphi}{\partial y})^2 = cts,$$

on the free surface $A^{'}CA$.

(3) $$\frac{\partial \varphi}{\partial y} = 0,$$

on the horizontal walls $A'B'$ and BA.

(4) $$\frac{\partial \varphi}{\partial x} = 0,$$

on the vertical wall $B'B$.

3. Exact solution

To solve the problem described in the previous section analytically, first we define a complex quantity Ω related to the velocity

(5) $$\Omega = \log(U\frac{dz}{df}) = \log(\frac{U}{q}) + i\theta.$$

Where q and θ are the modulus and argument of the velocity respectively. By this transformation, the flow field in the z-plane is transformed into a semi-infinite band in the Ω-plane (see Figure 3).

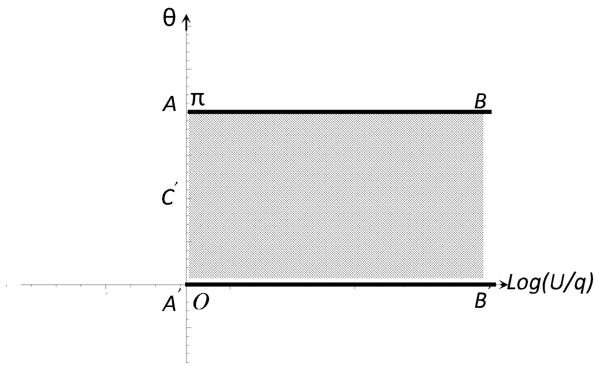

FIGURE 3. The Ω-plane

To transform the semi-infinite band in the Ω-plane to the upper half-plane of another complexes λ-plane, we utilize the Schwartz-Christoffel theorem, by respecting the direction and the orientation of the flow (see Figure 4). This transformation is given by:

(6) $$\lambda = \cosh\Omega.$$

The transformation which transforms the interior of the infinite band of the Ω-plane to the upper half-plan of the λ-plane is:

$$\lambda = \coth(\frac{\pi}{HU} f). \tag{7}$$

After calculations, we finds a relation between λ and z:

$$U\frac{dz}{d\lambda} = \frac{H}{\pi}(\frac{-2\lambda}{1-\lambda^2} - i\frac{2}{\sqrt{1-\lambda^2}}). \tag{8}$$

After integration of (8), with the choice of $z = 0$ into the point $\lambda = 0$. the exact form of the free surface is found as follows:

$$\begin{cases} x = \frac{1}{\pi}(\ln(1-\lambda^2) - 2\arcsin\lambda) \\ \\ y = \frac{1}{\pi}(-2\arcsin\lambda - \ln(1-\lambda^2)). \end{cases} \quad , \quad \text{for } -1 \leq \lambda \leq 1 \tag{9}$$

4. Numerical solution

To solve this problem numerically, we applied the series truncation technique. Following Birkhoff and Zarantonello and Vanden-Broeck we define a new variable t by the relation:

$$f = \frac{2}{\pi}\log\frac{1-t}{1+t} \tag{10}$$

This transformation maps the flow domain into the half of the unit disk in the complex t plane. So that the free surface is mapped on to the circumference $\overset{\frown}{A'CA}$ and the points B' and B are transformed into the points $t = -b$ and $t = b$ respectively (see Figure4).

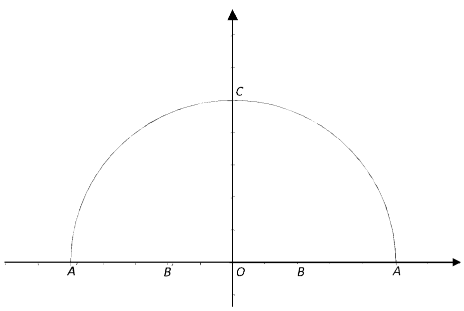

Figure 4. The t-plane

We introduce the function $\tau - i\theta$ as:

$$\zeta = \frac{df}{dz} = e^{\tau - i\theta}. \tag{11}$$

In these new variables (2) becomes:

$$e^{\tau} = 1, \tag{12}$$

The kinematic conditions (3)and(4)on the channel walls $A^{'}B^{'}$, $B^{'}B$ and BA can be expressed as:

$$\Im\zeta = 0 \qquad on \qquad -1 < t < -b \;\; or \;\; b < t < 1. \tag{13}$$

$$\Re\zeta = 0 \qquad on \qquad -b < t < b \tag{14}$$

We note that the flow is potential everywhere except at $t = \pm b$, where we have a flow in a corner and stagnant fluid. Hence local analysis is required. We note that

$$\zeta = \bigcirc(\frac{t-b}{1-b})^{\frac{1}{2}} \quad as \;\; t \longrightarrow b. \tag{15}$$

$$\zeta = \bigcirc(\frac{t+b}{1+b})^{\frac{1}{2}} \quad as \;\; t \longrightarrow -b. \tag{16}$$

We now have determined the local behavior of the flow near the the zero of the velocity, we seek $\zeta(t)$ in the form:

$$\zeta(t) = (\frac{t^2-b^2}{1-b^2})^{\frac{1}{2}} \times g(t). \tag{17}$$

Where the function $g(t)$ is bounded and continuous on the unit circle and analytic in the interior. The conditions (13) and (14) show that $g(t)$ can be expanded in the form of a Taylor expansion in even powers of t. Hence,

$$e^{\tau - i\theta} = \zeta(t) = (\frac{t^2-b^2}{1-b^2})^{\frac{1}{2}} e^{\sum_{n=1}^{\infty} a_n t^{2n}}. \tag{18}$$

By choosing all the coefficients a_n to be real, the function (18) satisfies (13) and (14). The coefficients a_n and b have to be determined to satisfy (12).

We use the notation $t = |t|\, e^{i\sigma}$ so that points on $A^{'}CA$ are given by

$$t = e^{i\sigma}, 0 < \sigma < \pi. \tag{19}$$

Using (19) we rewrite (12) in the form:

$$e^{\overline{\tau}(\sigma)} = \frac{(1+b^4-2b^2cos2\sigma)^{\frac{1}{4}}}{(1-b^2)^{\frac{1}{2}}} e^{\sum_{k=1}^{\infty} a_k cos(2(k-1)\sigma)} = 1 \tag{20}$$

We solve the problem approximately by truncating the infinite series in (19) after N terms. We find the N coefficients a_n by collocation. Thus we introduce the N mesh points

$$\sigma(I) = \frac{\pi}{N}(I - \frac{1}{2}), \qquad I = 1, \ldots, N+1 \tag{21}$$

Using (19) we obtain $[\overline{\tau}(\sigma)]_{\sigma=\sigma_I}$, in terms of coefficients a_n. Thus, we obtain N nonlinear algebraic equations of N unknowns $(a_{n,\;\; n=1,\ldots,N})$ for given value of the parameter b. We used the Newton method for solving this system of equations and MATLAB has also been used as a programming utility.

Finally, the shape of the free surface is obtained by integrating numerically the relation

$$\frac{\partial x}{\partial \sigma} = \frac{2}{\pi}\frac{1}{\sin(\sigma)}e^{-\bar{\tau}}\cos(\bar{\theta}) \tag{22}$$

and

$$\frac{\partial y}{\partial \sigma} = \frac{2}{\pi}\frac{1}{\sin(\sigma)}e^{-\bar{\tau}}\sin(\bar{\theta}) \tag{23}$$

Here $\bar{\tau}(\sigma)$ and $\bar{\theta}(\sigma)$ denote the values of τ and θ on the free surface $A^{'}CA$.

We used the numerical scheme described above to compute the approximate solutions for different values of the length of the vertical wall $B^{'}B$. The solutions for various values of b are obtained. Table 1 present some values of the coefficients a_n of the series (18) for different values of the parameter b. We also see the coefficients a_n decrease rapidly As n increases.

Table 1. Some values of coefficients a_k for various values of the parameter b

b	a_1	a_{20}	a_{40}
0.25	$-3,2269 \times 10^{-2}$	$1,7544 \times 10^{-13}$	$4,3360 \times 10^{-15}$
0.5	-0.14384	$3,0349 \times 10^{-13}$	$5,0909 \times 10^{-15}$
0.75	-0.4133	$4,7048 \times 10^{-7}$	$1,6224 \times 10^{-12}$

Most of the calculations were done and presented with $N = 40$.

In Figure 5, we present a comparison between the free surface profiles obtained by the numerical and the exact methods for $b = 0.25$.

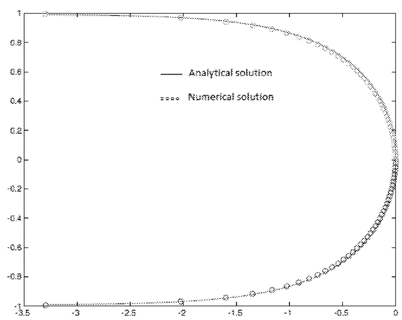

Figure 5. Comparison of the numerical free streamline shape with the exact theorical results for $b = 0.25$

For $b \to 1$ the length of the vertical wall $B^{'}B$ tend to infinity, witch we have a jet past a vertical wall in this case our result agree with that obtained by B. Bouderah, A. Gasmi and H. serguine [4] (see Figure 6)

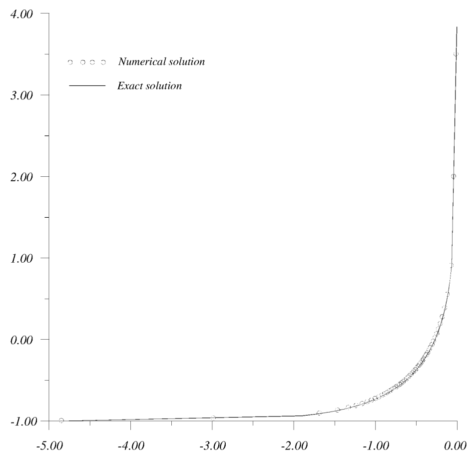

FIGURE 6. Comparison of the numerical free streamline shape with the exact theorical results for $b \to 1$

5. Conclusion

In this work, series truncation method and the free streamline theory methods have been successfully applied to solve two-dimensional steady flow problem in semi infinite chancel, these methods were used for free nonlinear boundary problem. The obtained results reveal that the proposed methods are very simple and straightforward, which reduce the propped problem to a one-dimensional problem. Furthermore the series truncation approach does not require discretization of the whole domain, but it is sufficient to descrizate the free surface only and therefore is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution.

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